REPRESENTABILITY FOR SOME MODULI STACKS OF FRAMED SHEAVES

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Introduction. Moduli problems for various kinds of framed sheaves have been studied and used in many settings (see, for example, [Tha94], [Bra91], [Nak94]), and there is a good general theory of moduli for semistable framed sheaves, thanks to the work of Huybrechts and Lehn ([HL95a], [HL95b]). By contrast, there seem to be only a few examples in which the *full* moduli functor for framed sheaves (without conditions of semistability) is known to be represented by a scheme. In this paper, we prove a representability theorem for the full moduli functors of framed torsion-free sheaves on projective surfaces under certain conditions.

Let S denote a smooth, connected complex projective surface, and let $D \subset S$ denote a smooth connected complete curve in S. Fix a vector bundle E on D. An *E-framed torsion-free sheaf on S* is a pair $(\mathcal{E}, \phi)$ consisting of a torsion-free sheaf $\mathcal{E}$ on S and an isomorphism $\phi : \mathcal{E}\big|_D \to E$; the isomorphism ϕ is called an *E-framing* of $\mathcal{E}$. An *isomorphism* of E-framed torsion-free sheaves on S is an isomorphism of the underlying torsion-free sheaves on S that is compatible with the framings. Let $\mathrm{TF}_S(E)$ denote the moduli functor for isomorphism classes of E-framed torsion-free sheaves on S. The reader should note that in the work of Huybrechts–Lehn the framing ϕ need *not* be an isomorphism; as a consequence of our more restrictive definition, the moduli functors that we study have no hope of being proper.

Suppose the vector bundle E satisfies

$$H^0\left(D, \operatorname{End} E \otimes N_{D/S}^{-k}\right) = 0 \tag{1}$$

for all $k \geq 1$; here $N_{D/S}$ is the normal bundle of D in S. If $D \subset S$ is an arbitrary curve, there may be very few such bundles. However, if D is smooth and has positive self-intersection in S, then $N_{D/S}^{-1}$ is a negative line bundle on D, and consequently this condition on E is an open condition which is satisfied by all semistable vector bundles on D.

Theorem 1. *Suppose that S is a smooth, connected complex projective surface and $D \subset S$ is a smooth connected complete curve. Suppose, in addition, that E is a vector bundle on D that satisfies Condition* (1) *for all $k \geq 1$. Then the functor $\mathrm{TF}_S(E)$ is represented by a scheme.*

In the proof of Theorem 1 we work in the slightly more general setting of a family of vector bundles on D, parametrized by a scheme U, that satisfies Condition (1) for all $k \geq 1$ at every point of U. Note also that the reader who is familiar with the language of stacks may restate Theorem 1 in the following form: over the substack of its target that parametrizes vector bundles on D that satisfy Condition (1) for all $k \geq 1$, the fibers of the restriction morphism from the moduli stack of torsion-free

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sheaves on S that are locally free along D to the moduli stack of vector bundles on D are schemes.

Functors of the type we study here arose naturally (in some special cases) in the representation-theoretic constructions of Nakajima; Theorem 1 demonstrates that the existence of the fine moduli schemes used by Nakajima is a much more general phenomenon, one which we hope can be exploited more widely in the study of sheaves on noncompact surfaces. The new ingredient in our proof of Theorem 1 is the use of formal geometry along the curve D; in particular, the techniques used here are completely different from those of [HL95a], [HL95b], and make no use of geometric invariant theory (GIT). Although Lehn ([Leh93]) has, under some conditions on the curve D and the bundle E along the curve, proven that the full moduli functors for vector bundles on S with framing along D by E are represented by *algebraic spaces*, from the point of view of the usual GIT techniques it is perhaps surprising that there is a fine moduli *scheme* (a much stronger fact) for all framed sheaves: indeed, there can be framed sheaves that are not semistable for *any* polarization.

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Affine bundles over $\mathrm{Bun}^\circ(D)$**.** In this section we construct the fundamental affine bundles $\mathbf{A}_n$ (for n in the range $1 \le n < \infty$) over U that we will use to embed the functor $\mathrm{TF}_S(E)$ in a scheme. The construction of these bundles and the description of the universal properties they possess must be well known (cf. [Gri66], in which the relevant cohomology groups are discussed), but the author does not know a suitable reference.

Fix a surface S, a curve D in S, a scheme U, and a vector bundle E on $D \times U$ as in Theorem 1. Let $D^{(n)}$ (that is, D with structure sheaf $\mathcal{O}_S/I_D^{n+1}$, $0 \le n < \infty$) denote the nth order neighborhood of D in S.

Definition 2. Let $\mathcal{A}_n$ denote the moduli functor over U of isomorphism classes of triples $(\mathcal{E}, V \xrightarrow{f} U, \phi)$ consisting of

1. a vector bundle $\mathcal{E}$ on $D^{(n)} \times V$,
2. a morphism $f : V \to U$, and
3. an isomorphism $\phi : \mathcal{E}\big|_{D\times V} \to (1_D \times f)^* E$.

Suppose that $\mathcal{E}$ is a vector bundle over $D^{(n)}$; then $\mathcal{E}$ has a canonical (decreasing) filtration as an $\mathcal{O}_{D^{(n)}}$-module with filtered pieces $F_j\mathcal{E} = I_D^j\mathcal{E}$, where I_D is the ideal of $D \subset D^{(n)}$. By its construction, this filtration is preserved by any endomorphism of the vector bundle $\mathcal{E}$, and moreover $F_j\mathcal{E}/F_{j+1}\mathcal{E} \cong N_{D/S}^{-j} \otimes \big(F_0\mathcal{E}/F_1\mathcal{E}\big)$ provided $0 \le j \le n$. Using these facts together with the exact sequence

$$0 \to \mathrm{Hom}(E, E \otimes N_{D/S}^{-n}) \to \mathrm{Hom}(\mathcal{E},\mathcal{E}) \to \mathrm{Hom}\left(\mathcal{E}, \mathcal{E}\big|_{D^{(n-1)}}\right) \to 0$$

and condition (1), one may prove by induction on n that $\mathrm{End}(\mathcal{E}) \subseteq \mathrm{End}(\mathcal{E}\big|_D)$ and consequently that E-framed bundles on $D^{(n)}$ are rigid.

Evidently $\mathcal{A}_0 \cong U$; moreover, there are maps $\pi_{n+1} : \mathcal{A}_{n+1} \to \mathcal{A}_n$ for all $n \geq 0$.

Proposition 3. *Each $\mathcal{A}_n$ $(n \geq 1)$ is represented by a scheme $\mathbf{A}_n$ that is an affine bundle over $\mathbf{A}_{n-1}$.*

Proof. Working inductively, it will suffice to construct an $\mathbf{A}_{n-1}$-scheme $\mathbf{A}_n$ that represents $\mathcal{A}_n$ and is an affine bundle over $\mathbf{A}_{n-1}$. Fix a universal bundle $E^{(n-1)}$ on $D^{(n-1)} \times \mathbf{A}_{n-1}$. For any scheme T, an element of $\mathcal{A}_n(T)$ determines a map $f : T \to \mathbf{A}_{n-1}$, and, if $(\mathcal{E}, \phi)$ is the given element of $\mathcal{A}_n(T)$, there is an isomorphism of $\mathcal{E}\big|_{D^{(n-1)} \times T}$ with $(1 \times f)^* E^{(n-1)}$ compatibly with the framings by E. But then, because E-framed bundles on $D^{(n-1)}$ are rigid, we find that $\mathcal{A}_n$ as a functor over $\mathbf{A}_{n-1}$ is isomorphic to the functor taking $f : T \to \mathbf{A}_{n-1}$ to the set of isomorphism classes of pairs $(\mathcal{E}, \phi)$ consisting of a bundle $\mathcal{E}$ on $D^{(n)} \times T$ together with an isomorphism ϕ of $\mathcal{E}\big|_{D^{(n-1)} \times T}$ with $(1 \times f)^* E^{(n-1)}$. We will refer to such a pair as an $E^{(n-1)}$-framed bundle.

Because the statement of the proposition is local on $\mathbf{A}_{n-1}$, we may assume that $\mathbf{A}_{n-1}$ is an affine scheme that is the spectrum of a local ring R. For simplicity, write $\mathcal{O} = \mathcal{O}_{D^{(n-1)} \times \mathbf{A}_{n-1}}$ and $\mathcal{O}' = \mathcal{O}_{D^{(n)} \times \mathbf{A}_{n-1}}$. The "change of rings" spectral sequence (see Chap. XVI, Section 5 of [CE56])

$$E_2^{p,q} = \operatorname{Ext}^p_{\mathcal{O}} \left(\underline{\operatorname{Tor}}_q^{\mathcal{O}'}(E^{(n-1)}, \mathcal{O}), E(-nD) \right) \Rightarrow \operatorname{Ext}^{p+q}_{\mathcal{O}'} \left(E^{(n-1)}, E(-nD) \right)$$

yields the exact sequence of terms of low degree

$$\begin{aligned}(2) \quad 0 \to \operatorname{Ext}^1_{\mathcal{O}} \left(E^{(n-1)}, E(-nD) \right) &\to \operatorname{Ext}^1_{\mathcal{O}'} \left(E^{(n-1)}, E(-nD) \right) \\ &\xrightarrow{\beta} \operatorname{Hom} \left(\underline{\operatorname{Tor}}_1^{\mathcal{O}'}(E^{(n-1)}, \mathcal{O}), E(-nD) \right) \to 0.\end{aligned}$$

Note that β is surjective since the next term in the sequence is $\operatorname{Ext}^2_{\mathcal{O}} \left(E^{(n-1)}, E(-nD) \right)$, which vanishes because D is one-dimensional. Using $\underline{\operatorname{Tor}}_1^{\mathcal{O}'}(E^{(n-1)}, \mathcal{O}) \cong E(-nD)$ one may check that there is a canonical element e of $\operatorname{Hom} \left(\underline{\operatorname{Tor}}_1^{\mathcal{O}'}(E^{(n-1)}, \mathcal{O}), E(-nD) \right)$ such that $\beta^{-1}(e)$ is exactly the $\operatorname{Ext}^1_{\mathcal{O}} \left(E^{(n-1)}, E(-nD) \right)$-subtorsor of $\operatorname{Ext}^1_{\mathcal{O}'} \left(E^{(n-1)}, E(-nD) \right)$ that classifies 1-extensions

$$0 \to E(-nD) \to \mathcal{E} \to E^{(n-1)} \to 0$$

for which $\mathcal{E}$ is a locally free $\mathcal{O}'$-module. Now, Condition (1), together with Cohomology and Base Change, implies that the R-module $\operatorname{Ext}^1_{\mathcal{O}} \left(E^{(n-1)}, E(-nD) \right) \cong H^1 \left(D \times \mathbf{A}_{n-1}, \operatorname{End}(E) \otimes N_{D/S}^{-n} \right)$ is projective, hence free. One can easily construct, moreover, a universal 1-extension over $D^{(n)} \times \mathbf{A}_{n-1} \times \beta^{-1}(e)$ (using, for example, an affine subspace of the Čech cocycles that maps isomorphically to $\beta^{-1}(e)$ to furnish gluing data). Because the exact sequence (2) and the element e are functorial under pullback along morphisms of affine schemes $\operatorname{Spec} R' \xrightarrow{f} \operatorname{Spec} R = \mathbf{A}_{n-1}$, this universal 1-extension induces a functorial bijection between the set $\beta_{R'}^{-1}(e)$ (the inverse image of the canonical element under the base-changed map β) and the set of isomorphism classes of pairs $(\mathcal{E}, \phi)$ consisting of a vector bundle $\mathcal{E}$ on $D^{(n)} \times \operatorname{Spec} R'$ and a framing $\phi : \mathcal{E}\big|_{D^{(n-1)} \times \operatorname{Spec} R'} \to (1 \times f)^* E^{(n-1)}$.

Consequently $\mathcal{A}_n$ is represented as a functor over $\mathbf{A}_{n-1}$ by the torsor over $\operatorname{Spec} \operatorname{Sym}^\bullet \operatorname{Ext}^1_{\mathcal{O}} \left(E^{(n-1)}, E(-nD) \right)$ defined by $\beta^{-1}(e)$, proving the proposition. $\square$

Proof of Theorem 1. There is a compatible family of morphisms $F_n : \mathrm{TF}_S(E) \to \mathbf{A}_n$ given by restriction. Fix a $\operatorname{Spec}\mathbf{C}$-valued point of $\mathrm{TF}_S(E)$, that is, a point $u \in U$ together with an E_u-framed pair $(\mathcal{F}, \phi)$ on S. We will show that there is an open subfunctor Z of $\mathrm{TF}_S(E)$ that contains $(\mathcal{F}, \phi)$ and is represented by a scheme.

Fix a polarization H of S, and choose m sufficiently large that

1. $\mathcal{F} \otimes H^m$ is globally generated and
2. $H^1(\mathcal{F} \otimes H^m) = H^2(\mathcal{F} \otimes H^m) = 0$.

Further, fix n sufficiently large that the restriction map

$$H^0(\mathcal{F} \otimes H^m) \to H^0\left(\mathcal{F} \otimes H^m\big|_{D^{(n)}}\right)$$

is injective; it is possible to choose such an n because $\mathcal{F}$ is torsion-free. Finally, choose m' sufficiently large that $H^1\left(\mathcal{F} \otimes H^{m+m'}\big|_{D^{(n)}}\right) = 0$.

Next, let $Z \subseteq \mathrm{TF}_S(E)$ denote the open subfunctor parametrizing those triples $\left(W \xrightarrow{f} U, \mathcal{E}, \phi : \mathcal{E}\big|_{D \times W} \to (1 \times f)^* E\right)$ for which the family $\mathcal{E}$ satisfies the following conditions:

a. $\mathcal{E}_w \otimes H^m$ is globally generated for all $w \in W$,
b. $H^1(\mathcal{E}_w \otimes H^m) = H^2(\mathcal{E}_w \otimes H^m) = 0$ for all $w \in W$,
c. the map $H^0(\mathcal{E}_w \otimes H^m) \to H^0\left(\mathcal{E}_w \otimes H^m\big|_{D^{(n)}}\right)$ is injective for all $w \in W$, and
d. $H^1\left(\mathcal{E}_w \otimes H^{m+m'}\big|_{D^{(n)}}\right) = 0$ for all $w \in W$.

In the previous section we showed that there is a universal vector bundle $E^{(n)}$ on $D^{(n)} \times \mathbf{A}_n$. Fix an element of $Z(W)$; then the map $F_n(W) : W \to \mathbf{A}_n$ yields a vector bundle $(1 \times F_n)^* E^{(n)}$ on $D^{(n)} \times W$ together with an isomorphism

$$\mathcal{E}_W\big|_{D^{(n)} \times W} \xrightarrow{\phi_n} (1 \times F_n)^* E^{(n)};$$

here $\mathcal{E}_W$ denotes the torsion-free sheaf on $S \times W$ determined by the fixed element of $Z(W)$. Let p_W denote the projection $S \times W \to W$. Then by construction the sheaves $(p_W)_* \mathcal{E}_W \otimes H^m$, $(p_W)_* \mathcal{E}_W \otimes H^{m+m'}$, and $(p_W)_*\left(\mathcal{E}_W \otimes H^{m+m'}\big|_{D^{(n)} \times W}\right)$ are vector bundles on W, and, choosing a section s of $H^{m'}$ the zero locus of which has transverse intersection with D, there is a commutative diagram

$$\begin{tikzcd}
(p_W)_* \mathcal{E}_W \otimes H^m \arrow[r] \arrow[d, "\otimes s"] & (p_W)_*\left(\mathcal{E}_W \otimes H^m\big|_{D^{(n)} \times W}\right) \arrow[d, "\otimes s"] \\
(p_W)_* \mathcal{E}_W \otimes H^{m+m'} \arrow[r] & (p_W)_*\left(\mathcal{E}_W \otimes H^{m+m'}\big|_{D^{(n)} \times W}\right)
\end{tikzcd}$$

for which the vertical arrows (given by tensoring with s) and the top row are injective. Using ϕ_n, we may replace this diagram canonically with the diagram

$$\begin{tikzcd}
(p_W)_*(\mathcal{E}_W \otimes H^m) \arrow[r] \arrow[d, "\otimes s"] & (p_W)_*\left((1 \times F_n)^* E^{(n)} \otimes H^m\right) \arrow[d, "\otimes s"] \\
(p_W)_* \mathcal{E}_W \otimes H^{m+m'} \arrow[r] & (p_W)_*\left((1 \times F_n)^* E^{(n)} \otimes H^{m+m'}\right).
\end{tikzcd}$$

Now, by assumption (d) on W, we have

$$(p_W)_*\left((1\times F_n)^*E^{(n)}\otimes H^{m+m'}\right) = F_n^*\left((p_{\mathbf{A}_n})_*(E^{(n)}\otimes H^{m+m'})\right),$$

where $p_{\mathbf{A}_n} : D^{(n)}\times\mathbf{A}_n \to \mathbf{A}_n$ is the projection, and so finally we obtain the diagram of vector bundles

$$\begin{tikzcd}
(p_W)_*\mathcal{E}_W\otimes H^m \arrow[d, "\otimes s"] \arrow[dr, "r"] & \\
(p_W)_*\mathcal{E}_W\otimes H^{m+m'} \arrow[r] & F_n^*\left((p_{\mathbf{A}_n})_*(E^{(n)}\otimes H^{m+m'})\right)
\end{tikzcd}$$

on W, where the diagonal map r and the map $\otimes s$ are injective. By construction, furthermore, the image of the morphism r is a vector subbundle of $F_n^*\left((p_{\mathbf{A}_n})_*(E^{(n)}\otimes H^{m+m'})\right)$ and consequently determines a morphism $W\to\mathbf{Gr}$ over $\mathbf{A}_n$, where $\mathbf{Gr}\xrightarrow{q}\mathbf{A}_n$ denotes the relative Grassmannian for the vector bundle $(p_{\mathbf{A}_n})_*(E^{(n)}\otimes H^{m+m'})$ on $\mathbf{A}_n$, the fiber of which over $a\in\mathbf{A}_n$ parametrizes vector subspaces of $H^0(E^{(n)}\otimes H^{m+m'})$ that are of dimension $h^0(\mathcal{F}\otimes H^m)$.

We now construct a Quot-scheme over $\mathbf{Gr}$ that we will use to represent Z. We may pull back $(p_{\mathbf{A}_n})_*(E^{(n)}\otimes H^{m+m'})$ to $\mathbf{Gr}$ to obtain a vector bundle $q^*(p_{\mathbf{A}_n})_*(E^{(n)}\otimes H^{m+m'})$ on (an open subset of) $\mathbf{Gr}$, with universal subbundle

$$\mathcal{U}\subset q^*(p_{\mathbf{A}_n})_*(E^{(n)}\otimes H^{m+m'})$$

of rank $h^0(\mathcal{F}\otimes H^m)$. If $p_{\mathbf{Gr}} : S\times\mathbf{Gr}\to\mathbf{Gr}$ denotes the projection to $\mathbf{Gr}$, we obtain a bundle $p_{\mathbf{Gr}}^*\mathcal{U}\subset p_{\mathbf{Gr}}^*q^*(p_{\mathbf{A}_n})_*(E^{(n)}\otimes H^{m+m'})$ on $S\times\mathbf{Gr}$, as well as a quotient

$$p_{\mathbf{Gr}}^*q^*(p_{\mathbf{A}_n})_*(E^{(n)}\otimes H^{m+m'})\to(1\times q)^*(E^{(n)}\otimes H^{m+m'})$$

and subquotient $(1\times q)^*(E^{(n)}\otimes H^m)\subset(1\times q)^*(E^{(n)}\otimes H^{m+m'})$ that are sheaves on $S\times\mathbf{Gr}$ supported on $D^{(n)}\times\mathbf{Gr}$.

Consider the relative Quot-scheme $q' : \mathrm{Quot}_{S\times\mathbf{Gr}/S}(p_{\mathbf{Gr}}^*\mathcal{U})\longrightarrow\mathbf{Gr}$ that parametrizes quotient sheaves for the family $p_{\mathbf{Gr}}^*\mathcal{U}$ on $S\times\mathbf{Gr}/S$. There is a universal quotient $(1\times q')^*p_{\mathbf{Gr}}^*\mathcal{U}\to\mathcal{Q}$ on $S\times\mathrm{Quot}_{S\times\mathbf{Gr}/S}$, giving a diagram

$$\begin{tikzcd}
(1\times q')^*p_{\mathbf{Gr}}^*\mathcal{U} \arrow[r] \arrow[d] & (1\times q')^*p_{\mathbf{Gr}}^*q^*(p_{\mathbf{A}_n})_*(E^{(n)}\otimes H^{m+m'}) \arrow[d] \\
\mathcal{Q} & (1\times qq')^*(E^{(n)}\otimes H^m)\subset(1\times qq')^*(E^{(n)}\otimes H^{m+m'}).
\end{tikzcd} \tag{3}$$

There is a closed subscheme of $\mathrm{Quot}_{S\times\mathbf{Gr}/S}$ (see the proof of Theorem 1.6 of [Ser86]) that represents the subfunctor of those quotients the kernels of which project to zero in $(1\times qq')^*(E^{(n)}\otimes H^{m+m'})$, and a closed subscheme $\mathcal{C}$ of that closed subscheme that represents the sub-subfunctor that parametrizes those quotients the images of which in $(1\times qq')^*(E^{(n)}\otimes H^{m+m'})$ actually lie in the subsheaf $(1\times qq')^*(E^{(n)}\otimes H^m)$. $\mathcal{C}$ then represents the functor of quotients of $p_{\mathbf{Gr}}^*\mathcal{U}$ that map to $(1\times qq')^*(E^{(n)}\otimes H^m)$—that

is, it is exactly the closed subscheme over which Diagram (3) extends to

$$
\begin{tikzcd}
(1\times q')^* p^*_{\mathbf{Gr}}\mathcal{U} \arrow[rr] \arrow[d] & & (1\times q')^* p^*_{\mathbf{Gr}} q^* (p_{\mathbf{A}_n})_* (E^{(n)} \otimes H^{m+m'}) \arrow[d] \\
\mathcal{Q} \arrow[r] & (1\times qq')^*(E^{(n)}\otimes H^m) \arrow[r, phantom, "\subset"] & (1\times qq')^*(E^{(n)}\otimes H^{m+m'}).
\end{tikzcd}
\tag{4}
$$

Restricting further to an open subscheme $\mathcal{C}^\circ$ of $\mathcal{C}$, we may assume that, over $\mathcal{C}^\circ$, the map $\mathcal{Q}\big|_{D^{(n)}\times\mathcal{C}^\circ} \to (1\times qq')^*(E^{(n)}\otimes H^m)$ is an isomorphism, that $\mathcal{Q}$ is a family of torsion-free sheaves on S, and that conditions (a) through (d) are satisfied.

By construction the morphism $W \to \mathbf{Gr}$ lifts to a morphism $W \to \mathcal{C}^\circ$; this construction thus determines a morphism of functors $Z \to \mathcal{C}^\circ$. Similarly, there is a forgetful morphism $\mathcal{C}^\circ \to Z$. Finally, it is clear from the construction that these two morphisms of functors are inverses of each other, as desired. $\square$

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